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Long-time shallow-water equations with a varying bottom

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We present and discuss new shallow-water equations that model the long-time effects of slowly varying bottom topography and weak hydrostatic imbalance on the vertically averaged horizontal velocity of an incompressible fluid possessing a free surface and moving under the force of gravity. We consider the regime where the Froude number ϵ is much smaller than the aspect ratio δ of the shallow domain. The new equations are obtained from the $\epsilon \to 0$ limit of the Euler equations (the rigid-lid approximation) at the first order of an asymptotic expansion in δ^2. These equations possess local conservation laws of energy and vorticity which reflect exact layer mean conservation laws of the three-dimensional Euler equations. In addition, they convect potential vorticity and have a Hamilton's principle formulation. We contrast them with the Green–Naghdi equations.

1. Introduction

In this paper we derive equations that model the long-time effects of slowly varying topography and weak hydrostatic imbalance on the incompressible motion of an inviscid fluid in a shallow basin. These equations arise at two successive levels of an asymptotic expansion. Their structure is analogous to that of the two-dimensional Euler equations for an ideal incompressible fluid in that they locally conserve some energy and vorticity, convect some potential vorticity, have a Kelvin circulation theorem, and possess a Hamilton's principle formulation.

We consider fluid contained in a basin by a uniform gravitational acceleration g and fixed vertical lateral boundaries (i.e. no sloping beaches). The horizontal spatial coordinate $\boldsymbol{x}$ is thereby confined to a fixed bounded domain $\mathscr{D}$ with boundary $\partial\mathscr{D}$. The vertical coordinate z is chosen so that the mean height of the fluid's free upper surface is at $z = 0$. The fixed bottom topography is given by $z = -b(\boldsymbol{x})$, where $b(\boldsymbol{x})$ is strictly positive over $\mathscr{D}$. The free upper surface is given by $z = h(\boldsymbol{x}, t)$, so that the total thickness of the fluid layer is $\eta(\boldsymbol{x}, t) = b(\boldsymbol{x}) + h(\boldsymbol{x}, t)$. Consistency with the definition of h requires that $h(\boldsymbol{x}, t) \geqslant -b(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathscr{D}$ and $t \geqslant 0$. Both $\partial\mathscr{D}$ and $b(\boldsymbol{x})$ are assumed to vary in $\boldsymbol{x}$ over distances L that are large compared to the mean depth B, thereby defining a small aspect ratio denoted as $\delta \equiv B/L$.

The fluid motion is taken to be governed by the three-dimensional Euler equations for incompressible flow. The fluid velocity is decomposed into its horizontal and vertical components, denoted as $\boldsymbol{u}$ and w respectively. We will consider only those

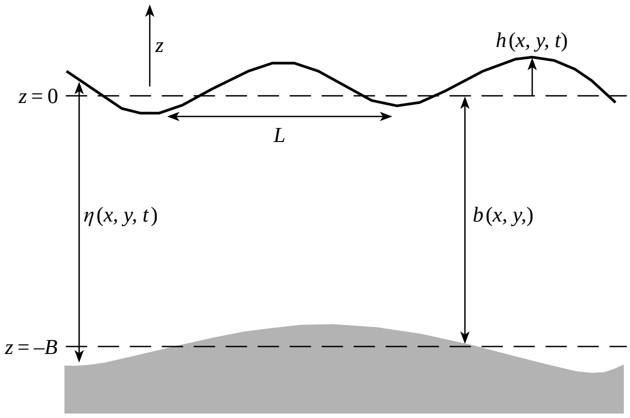

FIGURE 1. The set-up for the lake and great lake equations. Vertical scales have been exaggerated.

motions for which $\boldsymbol{u}$, w and h each vary in $\boldsymbol{x}$ over distances L, that is, we will make the long-wave approximation. In addition, we will assume that $\boldsymbol{u}$ has a characteristic magnitude U that is small compared to the gravity wave speed $(gB)^{1/2}$, thereby defining a small 'Froude number' denoted as $\epsilon \equiv U/(gB)^{1/2}$. For such motions w is smaller than U by a factor δ, while h is smaller than B by a factor ϵ^2. The situation is illustrated in figure 1.

Shallow-water equations arise in regimes where δ is small (cf. Pedlosky 1987, p. 57, or Whitham 1974, ch. 13). To leading order the fluid will then be in hydrostatic balance and the horizontal fluid velocity $\boldsymbol{u}$ can be assumed to be columnar – namely, independent of z. Rigid-lid equations arise in regimes where ϵ is small (cf. Allen, Newberger & Holman 1996). To leading order gravity waves will then be negligible. In regimes where both ϵ and δ are small the leading-order evolution of $\boldsymbol{u}(\boldsymbol{x}, t)$ and $h(\boldsymbol{x}, t)$ will be governed by equations that have the non-dimensional form

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla h = 0, \quad \nabla \cdot (b\boldsymbol{u}) = 0, \tag{1.1}$$

where ∇ is the horizontal gradient. Because these equations apply to a domain which is shallow compared to its width and whose free surface exhibits negligible surface motion, we call them the 'lake' equations. As we will see, the lake equations are quite robust, as they arise either as the long-wave approximation to the rigid-lid equations ($\epsilon \to 0$ first, then $\delta \to 0$) or as the small-Froude-number limit of the shallow-water equations ($\delta \to 0$ first, then $\epsilon \to 0$).

Our main concern will be regimes where $\epsilon \ll \delta$. We will carry out the δ-expansion of solutions of the rigid-lid equations to the next order, $O(\delta^2)$. At this order, the evolution of the vertically averaged horizontal fluid velocity, which we also denote by $\boldsymbol{u}(\boldsymbol{x}, t)$, and $h(\boldsymbol{x}, t)$ are found to be governed by new asymptotic equations, which we call the 'great lake' (GL) equations; these have the non-dimensional form

$$\partial_t \boldsymbol{v} + \boldsymbol{u} \cdot \nabla \boldsymbol{v} + (\nabla \boldsymbol{u})\boldsymbol{v} + \nabla\left(h - \tfrac{1}{2}|\boldsymbol{u}|^2\right) = 0, \qquad \nabla \cdot (b\boldsymbol{u}) = 0. \tag{1.2}$$

Here $(\nabla \boldsymbol{u})\boldsymbol{v} = \sum_{j=1}^{2} v_j \nabla u^j$. The GL equations are written in terms of the auxiliary field $\boldsymbol{v}(\boldsymbol{x}, t)$ defined by

$$\boldsymbol{v} = \boldsymbol{u} + \tfrac{1}{6}\delta^2 b^2 \nabla(\nabla \cdot \boldsymbol{u}). \tag{1.3}$$

Subject to the divergence condition $\nabla \cdot (b\boldsymbol{u}) = 0$, we show that relation (1.3) defines a certain positive-definite (and hence invertible) operator which relates $\boldsymbol{u}$ to $\boldsymbol{v}$. The invertibility ensures that $\boldsymbol{u}$ depends continuously on $\boldsymbol{v}$. The curl of $\boldsymbol{v}$ turns out to model the vertical average of a particular z-dependent component of the three-dimensional Euler vorticity. In particular, for motion originating from rest, which is necessarily irrotational, we show that this curl, and *not* the curl of $\boldsymbol{u}$, must vanish.

Notice that when the bottom is flat ($\nabla b = 0$) both the lake and GL equations reduce to the two-dimensional incompressible Euler equations with h playing the role of pressure. Indeed, we shall see that these equations share many structural similarities with the two-dimensional incompressible Euler equations. For example, in each case h is determined from an elliptic problem that arises from preservation of the (weighted) divergence condition.

Section 2 contains the derivation of the above equations starting from the three-dimensional Euler equations dimensionally rescaled in terms of δ and ϵ. We assume that the Froude number is much smaller than the aspect ratio ($\epsilon \ll \delta$) and obtain the so-called rigid-lid approximation by keeping only the leading-order terms in ϵ. Solutions of these equations are then expanded asymptotically in δ^2. The lake equations (1.1) arise at the leading-order while the GL equations (1.2) arise at $O(\delta^2)$. Section 3 shows how solutions of the GL equations obey local conservation laws for energy and vorticity. Moreover, solutions of the GL equations are shown to convect a potential vorticity. Section 4 shows how these structures reflect similar laws for vertically averaged quantities that are obeyed by solutions of the original Euler equations, including following from Hamilton's principle. Section 5 shows that the GL equations can also be understood as the $\epsilon \to 0$ limit of the shallow-water equations studied by Green & Naghdi (1976). The Green–Naghdi equations retain finite-amplitude gravity waves and their associated Boussinesq-type dispersion properties, which are removed in the GL equations by taking the $\epsilon \to 0$ limit.

2. Derivation of the model equations

2.1. *The scaled Euler equations in three dimensions*

The motion of an inviscid, incompressible fluid in the basin described above is governed by the three-dimensional Euler equations. We form non-dimensional variables in terms of the following natural units: ρ, the mass density; B, the mean depth; and $(gB)^{1/2}$, the gravity wave speed. Because the horizontal length scales are assumed to be long compared to B by the inverse aspect ratio $1/\delta$ where $\delta \ll 1$, we introduce non-dimensional spatial variables (denoted with asterisks) by

$$\boldsymbol{x} = \frac{1}{\delta} B\boldsymbol{x}^*, \quad z = Bz^*. \tag{2.1}$$

Because the horizontal fluid speed is assumed to be small compared to the gravity wave speed $(gB)^{1/2}$ by a factor of the Froude number ϵ where $\epsilon \ll 1$, and bearing in mind the aspect ratio, we introduce non-dimensional horizontal and vertical velocity fields by

$$\boldsymbol{u} = \epsilon (gB)^{1/2} \boldsymbol{u}^*, \quad w = \delta\epsilon (gB)^{1/2} w^*. \tag{2.2}$$

We will work with the modified pressure p (see Batchelor 1967, p. 176), which is related to the total pressure p_{tot} by $p_{tot} = p - \rho g z$; that is, p removes the hydrostatic part of the pressure $-\rho g z$ arising from the trivial static solution. We introduce the non-dimensional surface elevation h^* and (modified) pressure field p^* by

$$h = \epsilon^2 B h^*, \quad p = \epsilon^2 \rho g B p^*. \tag{2.3}$$

Finally, we introduce a non-dimensional temporal variable by

$$t = \frac{1}{\delta\epsilon}\left(\frac{B}{g}\right)^{1/2} t^* . \tag{2.4}$$

This is the time scale for a fluid parcel to traverse a typical horizontal length, which is longer than that for a gravity wave to traverse the same length by $O(1/\epsilon)$. Thus, the scaling in equations (2.1)–(2.4) selects shallow-water motions that have surface fluctuations whose amplitudes are small of $O(\epsilon^2)$ compared to the mean depth and consequently evolve on time scales which are long of $O(1/\epsilon)$ compared to the time scales of gravity wave evolution.

In the scaled variables (2.1)–(2.4) the non-dimensional form of the motion equations becomes (after dropping the asterisks)

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + w\partial_z\boldsymbol{u} + \nabla p = 0\,, \tag{2.5}$$

$$\delta^2\left(\partial_t w + \boldsymbol{u}\cdot\nabla w + w\partial_z w\right) + \partial_z p = 0\,. \tag{2.6}$$

These non-dimensional equations retain their dimensional form except that the vertical acceleration acquires a factor of δ^2. The non-dimensional form of the incompressibility condition acquires no small parameters and is

$$\nabla\cdot\boldsymbol{u} + \partial_z w = 0\,. \tag{2.7}$$

The dimensionless boundary conditions acquire factors of the small parameter ϵ^2 as follows. On the free surface we neglect surface tension and set the total pressure to zero, whereby the modified pressure p satisfies the dynamical boundary condition

$$p = h \quad \text{for } \boldsymbol{x}\in\mathscr{D} \text{ and } z = \epsilon^2 h(\boldsymbol{x},t)\,. \tag{2.8}$$

We also assume that no fluid crosses the boundaries (i.e. that the normal fluid velocity equals the interface velocity), which gives the kinematic boundary conditions

$$w = \epsilon^2\left(\partial_t h + \boldsymbol{u}\cdot\nabla h\right) \quad \text{for } \boldsymbol{x}\in\mathscr{D} \text{ and } z = \epsilon^2 h(\boldsymbol{x},t)\,, \tag{2.9}$$

$$w = -\boldsymbol{u}\cdot\nabla b \quad \text{for } \boldsymbol{x}\in\mathscr{D} \text{ and } z = -b(\boldsymbol{x})\,, \tag{2.10}$$

$$\widehat{\boldsymbol{n}}\cdot\boldsymbol{u} = 0 \quad \text{for } \boldsymbol{x}\in\partial\mathscr{D} \text{ and } -b(\boldsymbol{x}) < z < \epsilon^2 h(\boldsymbol{x},t)\,, \tag{2.11}$$

where $\widehat{\boldsymbol{n}}$ is the outward unit normal on $\partial\mathscr{D}$. Because the total volume of water remains fixed, we adopt the normalization

$$\int_{\mathscr{D}} h(\boldsymbol{x},t)\,\mathrm{d}x\mathrm{d}y = 0\,, \tag{2.12}$$

which states that the mean level of the upper water surface is fixed at $z = 0$.

In subsequent sections, we are going to make extensive use of vertical averages, or layer means, of dependent variables. For any function $f(\boldsymbol{x},z,t)$, the corresponding layer mean will be denoted by $\overline{f}$. In the context of the free upper surface $\overline{f}$ is given by

$$\overline{f}(\boldsymbol{x},t) \equiv \frac{1}{\eta(\boldsymbol{x},t)}\int_{-b(\boldsymbol{x})}^{\epsilon^2 h(\boldsymbol{x},t)} f(\boldsymbol{x},z,t)\,\mathrm{d}z\,, \tag{2.13}$$

where in the rescaled variables $\eta = b + \epsilon^2 h$. The layer mean of a convective derivative satisfies the useful identity (cf. Wu 1981)

$$\eta\,\overline{\left(\partial_t f + \boldsymbol{u}\cdot\nabla f + w\partial_z f\right)} = \partial_t\left(\eta\,\overline{f}\right) + \nabla\cdot\left(\eta\,\overline{\boldsymbol{u}f}\right), \tag{2.14}$$

as one can show via integration by parts and using the kinematic boundary conditions (2.9) and (2.10).

By choosing $f = 1$ and $f = \boldsymbol{u}$ in (2.14) and using (2.7) and (2.5) respectively, we obtain

$$\partial_t \eta + \nabla \cdot (\eta \overline{\boldsymbol{u}}) = 0\,, \tag{2.15}$$

and

$$\partial_t(\eta\overline{\boldsymbol{u}}) + \nabla \cdot (\eta \overline{\boldsymbol{u}\boldsymbol{u}}) = -\eta \overline{\nabla p}\,. \tag{2.16}$$

The first equation is a local conservation law in terms of the vertical average of $\boldsymbol{u}$ which corresponds to mass conservation for the three-dimensional flow. Shallow-water models can generally be viewed as arising from the balance equation (2.16) through a closure scheme based on a long-wave approximation for expressing $\overline{\boldsymbol{u}\boldsymbol{u}}$ and $\overline{\nabla p}$ in terms of η and $\overline{\boldsymbol{u}}$.

2.2. *Leading-order models*

In this article we shall consider a long-wave approximation, but in the regime of *very* small surface height amplitudes – that is, in the regime

$$\epsilon \ll \delta \ll 1\,. \tag{2.17}$$

Hence, we shall first consider the non-dimensional equations (2.5) – (2.11) in the small-Froude-number limit of $\epsilon \to 0$ while holding δ fixed and afterwards consider δ small. Because ϵ does not appear in the non-dimensional equations (2.5)–(2.7), these equations remain unchanged as $\epsilon \to 0$. However, ϵ does appear in the rescaled boundary conditions at $O(\epsilon^2)$. Upon letting $\epsilon \to 0$, the dynamic boundary condition (2.8) becomes

$$p = h \quad \text{for } \boldsymbol{x} \in \mathscr{D} \text{ and } z = 0\,, \tag{2.18}$$

while the kinematic boundary conditions (2.9) – (2.11) become

$$w = 0 \qquad \text{for } \boldsymbol{x} \in \mathscr{D} \text{ and } z = 0\,, \tag{2.19}$$

$$w = -\boldsymbol{u} \cdot \nabla b \quad \text{for } \boldsymbol{x} \in \mathscr{D} \text{ and } z = -b(\boldsymbol{x})\,, \tag{2.20}$$

$$\widehat{\boldsymbol{n}} \cdot \boldsymbol{u} = 0 \qquad \text{for } \boldsymbol{x} \in \partial\mathscr{D} \text{ and } -b(\boldsymbol{x}) < z < 0\,. \tag{2.21}$$

The system of equations (2.5)–(2.7) with boundary conditions (2.18)–(2.21) is called the 'rigid lid' approximation because the horizontal velocity behaves as if the top surface were fixed at its mean value. However, the designation 'rigid' may be misleading here because the upper surface dynamics is recovered from the dynamic boundary condition (2.18). The rigid-lid approximation removes gravity waves from the problem; this can be seen from the linearization of (2.5)–(2.7) about the trivial solution. (The boundary conditions (2.18)–(2.21) are already linear.)

In the context of the rigid-lid approximation the layer mean corresponding to a function $f(\boldsymbol{x}, z, t)$ is given by

$$\overline{f}(\boldsymbol{x}, t) = \frac{1}{b(\boldsymbol{x})} \int_{-b(\boldsymbol{x})}^{0} f(\boldsymbol{x}, z, t)\, \mathrm{d}z\,. \tag{2.22}$$

This is consistent with replacing η with b and setting the upper limit of integration to zero in (2.13). The layer mean of a convective derivative satisfies the useful identity

$$\overline{(\partial_t f + \boldsymbol{u} \cdot \nabla f + w \partial_z f)} = \partial_t \overline{f} + \frac{1}{b} \nabla \cdot \left(b\, \overline{\boldsymbol{u} f} \right), \tag{2.23}$$

as one can show directly via integration by parts and using the kinematic boundary conditions (2.19) and (2.20), which is also consistent with replacing η with b in (2.14).

By choosing $f = 1$ and $f = \boldsymbol{u}$ in (2.23) and using (2.7) and (2.5) respectively, we obtain

$$\nabla \cdot (b\overline{\boldsymbol{u}}) = 0, \tag{2.24}$$

and

$$\partial_t \overline{\boldsymbol{u}} + \frac{1}{b}\nabla \cdot (b\overline{\boldsymbol{u}\boldsymbol{u}}) = -\overline{\nabla p}. \tag{2.25}$$

These are consistent with replacing η with b in (2.15) and (2.16), which considerably simplifies those equations. The balance equation (2.25) will be used below, where we will devise a closure scheme based on a long-wave approximation for expressing $\overline{\boldsymbol{u}\boldsymbol{u}}$ and $\overline{\nabla p}$ in terms of $\overline{\boldsymbol{u}}$.

The incompressibility condition (2.7) can be integrated in z subject to the top kinematic boundary condition (2.19) to express w in terms of $\boldsymbol{u}$ as

$$w(\boldsymbol{x}, z, t) = \int_z^0 \nabla \cdot \boldsymbol{u}(\boldsymbol{x}, z_1, t)\, \mathrm{d}z_1. \tag{2.26}$$

Then the bottom kinematic boundary condition (2.20) will also be satisfied, provided $\boldsymbol{u}$ satisfies the divergence condition (2.24). Thus, after first eliminating w by (2.26) and then eliminating p in favour of $\boldsymbol{u}$ and h by integrating (2.6) in z subject to the dynamic boundary condition (2.18), we can consider (2.5) and (2.24) as equations for $\boldsymbol{u}$ and h subject to the boundary condition (2.21).

We now determine the leading-order behaviour of $\boldsymbol{u}$, w, p and h by formally passing to the $\delta \to 0$ limit in the motion equations (2.5) and (2.6). The w-equation (2.6) gives the hydrostatic balance condition $\partial_z p = 0$ which, upon applying the dynamic boundary condition (2.18), relates the pressure to the surface height as $p = h(\boldsymbol{x}, t)$. Hence, the horizontal motion equation (2.5) becomes

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + w\partial_z \boldsymbol{u} + \nabla h = 0. \tag{2.27}$$

After using (2.26) to eliminate w in favour of $\boldsymbol{u}$, this equation combines with the divergence condition (2.24) and the lateral boundary condition (2.21) to determine $\boldsymbol{u}$ and h. Considered as a system, (2.21), (2.24), (2.26) and (2.27) constitute the low-Froude-number limit of the long-wave equations studied by Benny (1973) and Zakharov (1981).

A considerable simplification is achieved by observing that the horizontal motion equation (2.27) may be satisfied by taking $\boldsymbol{u}$ to be columnar ($\boldsymbol{u} = \overline{\boldsymbol{u}}$), so that, like the pressure, it satisfies

$$\partial_z \boldsymbol{u} = 0. \tag{2.28}$$

Because its lateral boundary condition (2.21) contains no explicit z-dependence, the motion equation (2.27) will ensure that no z-dependence will develop in the leading-order horizontal velocity provided the initial state is z-independent. In this case the horizontal motion equation (2.27) and divergence condition (2.24) reduce to the system (1.1) over the horizontal domain $\mathscr{D}$, while the lateral boundary condition (2.21) reduces to simply $\widehat{\boldsymbol{n}} \cdot \boldsymbol{u} = 0$ for $\boldsymbol{x} \in \partial\mathscr{D}$. Given a solution $\boldsymbol{u}$ and h of (1.1) subject to this condition, we may recover the leading-order vertical velocity w from (2.26) as

$$w = -z\nabla \cdot \boldsymbol{u}, \tag{2.29}$$

and the leading-order pressure p from hydrostatic balance as $p = h$.

The lake equations (1.1) could also have been derived by first considering the non-dimensional equations (2.5)–(2.11) in the limit of $\delta \to 0$ while holding ϵ fixed and afterwards take the $\epsilon \to 0$ limit. In this case the first limit corresponds to the usual long-wave (hydrostatic) approximation and, upon assuming the horizontal fluid velocity is columnar, leads to the shallow-water equations:

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla h = 0, \quad \epsilon^2 \partial_t h + \nabla \cdot \left[\left(b + \epsilon^2 h \right) \boldsymbol{u} \right] = 0, \tag{2.30}$$

over the domain $\mathscr{D}$ with the boundary condition $\widehat{\boldsymbol{n}} \cdot \boldsymbol{u} = 0$. Passing to the $\epsilon \to 0$ limit in these equations, one sees that the equation for h in (2.30) becomes the divergence condition $\nabla \cdot (b\boldsymbol{u}) = 0$ while the motion equation for $\boldsymbol{u}$ remains unchanged.

The lake equations (1.1) are similar to the two-dimensional incompressible Euler equations of fluid dynamics. Indeed, the motion equation in (1.1) is identical to the Euler motion equation with the role of the pressure being played by the free-surface height h. The divergence condition in (1.1) differs from the Euler incompressibility condition in that it is weighted by the depth $b(\boldsymbol{x})$. The absence of gravity waves for the lake equations is analogous to the absence of acoustic waves for the incompressible Euler equations.

2.3. *Next-order model*

Recalling our basic scaling (2.17), we now seek solutions of the rigid-lid approximation up to order δ^2. To do so, we expand $\boldsymbol{u}$, w, p and h in powers of δ^2 as

$$\left. \begin{aligned} \boldsymbol{u} &= \boldsymbol{u}^{(0)} + \delta^2 \boldsymbol{u}^{(1)} + O(\delta^4), \qquad & w &= w^{(0)} + \delta^2 w^{(1)} + O(\delta^4), \\ p &= p^{(0)} + \delta^2 p^{(1)} + O(\delta^4), \qquad & h &= h^{(0)} + \delta^2 h^{(1)} + O(\delta^4), \end{aligned} \right\} \tag{2.31}$$

where $\boldsymbol{u}^{(0)}$ and $h^{(0)}$ solve the lake equations (1.1), $w^{(0)}$ is determined by (2.29), and $p^{(0)} = h^{(0)}$. These expansions will be put into equations (2.5)–(2.7) and boundary conditions (2.18)–(2.21) and matched up to order δ^2.

At $O(\delta^2)$ the vertical motion equation (2.6) yields

$$\begin{aligned} \partial_z p^{(1)} &= -\partial_t w^{(0)} - \boldsymbol{u}^{(0)} \cdot \nabla w^{(0)} - w^{(0)} \partial_z w^{(0)} \\ &= z \left[\left(\partial_t + \boldsymbol{u}^{(0)} \cdot \nabla \right) \left(\nabla \cdot \boldsymbol{u}^{(0)} \right) - \left(\nabla \cdot \boldsymbol{u}^{(0)} \right)^2 \right] \equiv z \; p_2^{(1)}. \end{aligned} \tag{2.32}$$

This equation can be integrated using the dynamic boundary condition (2.18) to find that the pressure at $O(\delta^2)$ is expressible as

$$p^{(1)} = h^{(1)}(\boldsymbol{x}, t) + \tfrac{1}{2} z^2 p_2^{(1)}(\boldsymbol{x}, t), \tag{2.33}$$

where $p_2^{(1)}$ is given in terms of $\boldsymbol{u}^{(0)}$ by equation (2.32). This order-δ^2 pressure contribution shows the effects of a weakly broken hydrostatic pressure balance.

We are now in a position to compute the approximate closure of the balance equation (2.25). First, because of the columnar motion assumption (2.28) at leading order, we notice that

$$\overline{\boldsymbol{u}} = \boldsymbol{u}^{(0)} + \delta^2 \overline{\boldsymbol{u}^{(1)}} + O(\delta^4), \tag{2.34}$$

and so

$$\overline{\boldsymbol{u}\boldsymbol{u}} = \overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}} + O(\delta^4). \tag{2.35}$$

Second, through (2.33) and by using (2.28) again, the vertical average of ∇p becomes

$$\overline{\nabla p} = \nabla h + \delta^2 \tfrac{1}{6} b^2 \nabla p_2^{(1)} + O(\delta^4). \tag{2.36}$$

The $\nabla p_2^{(1)}$ term above can be expressed explicitly in terms of $\bar{\boldsymbol{u}}$ through (2.32) and (2.34) as

$$\tfrac{1}{6}b^2\nabla p_2^{(1)} = \Big(\partial_t + \bar{\boldsymbol{u}}\cdot\nabla + \Big(\nabla\bar{\boldsymbol{u}}\Big)\Big)\Big(\tfrac{1}{6}b^2\nabla\Big(\nabla\cdot\bar{\boldsymbol{u}}\Big)\Big) + O(\delta^2). \tag{2.37}$$

Third, boundary condition (2.21) implies that $\bar{\boldsymbol{u}}$ identically satisfies

$$\hat{\boldsymbol{n}}\cdot\bar{\boldsymbol{u}} = 0 \quad \text{for } \boldsymbol{x}\in\partial\mathscr{D}. \tag{2.38}$$

Finally, by inserting (2.35) and (2.37) in (2.25) and using (2.24), one easily checks that (2.25) reduces to

$$\partial_t\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}\cdot\nabla\bar{\boldsymbol{u}} + \nabla h + \delta^2\Big(\partial_t + \bar{\boldsymbol{u}}\cdot\nabla + (\nabla\bar{\boldsymbol{u}})\Big)\Big(\tfrac{1}{6}b^2\nabla(\nabla\cdot\bar{\boldsymbol{u}})\Big) = O(\delta^4). \tag{2.39}$$

Although no assumptions were made concerning the vertical dependence of the correction $\boldsymbol{u}^{(1)}$ to $\boldsymbol{u}^{(0)}$, the result (2.39) is a closed equation for the vertical average of the horizontal velocity $\bar{\boldsymbol{u}}$ up to $O(\delta^2)$.

Upon introducing the auxiliary variable $\bar{\boldsymbol{v}}$ by

$$\bar{\boldsymbol{v}} \equiv \bar{\boldsymbol{u}} + \delta^2\tfrac{1}{6}b^2\nabla(\nabla\cdot\bar{\boldsymbol{u}}), \tag{2.40}$$

the left-hand side of equation (2.39) takes the more compact form

$$\partial_t\bar{\boldsymbol{v}} + \bar{\boldsymbol{u}}\cdot\nabla\bar{\boldsymbol{v}} + (\nabla\bar{\boldsymbol{u}})\bar{\boldsymbol{v}} + \nabla\Big(h - \tfrac{1}{2}|\bar{\boldsymbol{u}}|^2\Big) = O(\delta^4). \tag{2.41}$$

Exploiting the formal smallness of the $O(\delta^4)$ term, we now designate by $\boldsymbol{u}$ and h the solution of system (2.24), (2.38), and (2.41) with this term replaced by zero, so that $\boldsymbol{u}$ and h satisfy the GL equations (1.2) with the definition (1.3) presented in the Introduction.

To restore the GL equations (1.2)–(1.3) to dimensional form, one simply removes the δ^2 in the definition of $\boldsymbol{v}$ (1.3) and replaces h with gh in the motion equation (1.2), yielding

$$\partial_t\boldsymbol{v} + \boldsymbol{u}\cdot\nabla\boldsymbol{v} + (\nabla\boldsymbol{u})\boldsymbol{v} + \nabla\Big(gh - \tfrac{1}{2}|\boldsymbol{u}|^2\Big) = 0, \quad \nabla\cdot(b\boldsymbol{u}) = 0. \tag{2.42}$$

The boundary condition $\hat{\boldsymbol{n}}\cdot\boldsymbol{u} = 0$ for $\boldsymbol{x}\in\partial\mathscr{D}$ remains unchanged when dimensions are restored. It is important to note when considering initial data for (2.42) (or even the lake equations) that care must be taken to be sure the data are consistent with the scalings introduced in the last Section, otherwise the resulting solutions could lead to unphysical results such as $h < -b$ (cf. Camassa, Holm & Levermore 1996 for further discussion).

The GL equations (1.2) reduce to the lake equations (1.1) in the limit $\delta\to 0$. The $O(\delta^2)$ difference between these equations is due to the combined effects of bottom topography and hydrostatic imbalance. We emphasize that bottom topography and hydrostatic imbalance play a combined role. If the bottom is flat, so that ∇b vanishes, then $\nabla\cdot\boldsymbol{u} = 0$ and $\boldsymbol{v}$ reduces to $\boldsymbol{u}$ with no correction for hydrostatic imbalance. In this case, the second-order derivative terms in the definition of $\boldsymbol{v}$ in (1.2) vanish entirely, and the lake equations and the GL equations both reduce to the Euler equations for two-dimensional incompressible inviscid flow. Thus, the effects of bottom topography and hydrostatic pressure imbalance are crucially coupled in the asymptotic shallow-water expansion (2.31) in the small surface deviation limit, with $\epsilon = o(\delta)$. Notice that in this ordering $\epsilon = o(\delta)$, the GL equations offer a $o(\delta^2)$ approximation to the original free-surface problem for the Euler equations. We also emphasize that these coupled non-hydrostatic and topographic effects are not to be confused with ordinary

Boussinesq-type dispersion of gravity waves, which do not exist in this model (cf. §5, where the comparison with gravity wave dispersion is discussed further.) We shall analyse the lake and GL systems in the remainder of the paper.

3. Structure of the GL equations

3.1. *Elliptic equation for the free-surface elevation*

For those $\boldsymbol{u}$ that satisfy $\nabla \cdot (b\boldsymbol{u}) = 0$, relation (1.3) is equivalent to

$$b\boldsymbol{v} = b\boldsymbol{u} + \Big[-\tfrac{1}{3}\nabla\Big(b^3\nabla\cdot\boldsymbol{u}\Big) - \tfrac{1}{2}\nabla\Big(b^2\boldsymbol{u}\cdot\nabla b\Big) + \tfrac{1}{2}b^2(\nabla\cdot\boldsymbol{u})\nabla b + b(\boldsymbol{u}\cdot\nabla b)\nabla b\Big]$$
$$\equiv \mathscr{L}(b)\boldsymbol{u}. \tag{3.1}$$

The operator $\mathscr{L}(b)$ defined above is self-adjoint and positive-definite (and therefore invertible) because $b > 0$. Its invertibility ensures that $\boldsymbol{u}$ depends continuously on $\boldsymbol{v}$ (cf. Levermore, Oliver & Titi 1996*a*, *b* for further discussion). In particular, the operator $\mathscr{L}(b)$ arises in the solution for h from the $\boldsymbol{v}$ equation (2.42). By operating on this equation with $\nabla\cdot b\mathscr{L}^{-1}(b)b$ and using the weighted divergence condition, an elliptic equation is obtained for h which is reminiscent of the Poisson equation for the pressure in the case of the Euler equations, namely

$$\nabla\cdot\Big[b\,\mathscr{L}(b)^{-1}\Big(b\Big[\nabla\Big(gh-\tfrac{1}{2}|\boldsymbol{u}|^2\Big)+\boldsymbol{u}\cdot\nabla\boldsymbol{v}+(\nabla\boldsymbol{u})\boldsymbol{v}\Big]\Big)\Big]=0. \tag{3.2}$$

The corresponding lateral boundary condition is

$$\widehat{\boldsymbol{n}}\cdot\Big[b\,\mathscr{L}(b)^{-1}\Big(b\Big[\nabla\Big(gh-\tfrac{1}{2}|\boldsymbol{u}|^2\Big)+\boldsymbol{u}\cdot\nabla\boldsymbol{v}+(\nabla\boldsymbol{u})\boldsymbol{v}\Big]\Big)\Big]=0 \quad \text{on} \quad \partial\mathscr{D}, \tag{3.3}$$

where $\widehat{\boldsymbol{n}}$ is the outward unit normal of the boundary $\partial\mathscr{D}$. Together, (3.2) and (3.3) determine h up to an additive constant that is fixed by the h-normalization (2.12).

3.2. *Conservation of energy, vorticity and circulation*

The great lake (GL) equations (2.42)–(3.1) possess several fundamental physical properties which combine to make them particularly nice theoretically.

First, the GL equations possess a local energy conservation law. Indeed, one verifies from (3.1) by direct calculation that

$$\partial_t\Big[b\Big(\tfrac{1}{2}|\boldsymbol{u}|^2+\tfrac{1}{6}\big(\boldsymbol{u}\cdot\nabla b\big)^2\Big)\Big]=b\,\boldsymbol{u}\cdot\partial_t\boldsymbol{v}-\tfrac{1}{6}\nabla\cdot\big(b^3\boldsymbol{u}\partial_t(\nabla\cdot\boldsymbol{u})\big). \tag{3.4}$$

Upon using (2.42) to eliminate $\partial_t\boldsymbol{v}$ above, one finds

$$\partial_t\Big[b\Big(\tfrac{1}{2}|\boldsymbol{u}|^2+\tfrac{1}{6}\big(\boldsymbol{u}\cdot\nabla b\big)^2\Big)\Big]+\nabla\cdot\Big[b\,\boldsymbol{u}\Big(gh-\tfrac{1}{2}|\boldsymbol{u}|^2+\boldsymbol{u}\cdot\boldsymbol{v}\Big)+\tfrac{1}{6}\big(b^3\boldsymbol{u}\partial_t(\nabla\cdot\boldsymbol{u})\big)\Big]=0. \tag{3.5}$$

The GL equations therefore conserve the positive-definite quadratic functional

$$E_{GL}=\int b\left(\tfrac{1}{2}\,|\boldsymbol{u}|^2+\tfrac{1}{6}\,(\boldsymbol{u}\cdot\nabla b)^2\right)\,\mathrm{d}x\mathrm{d}y, \tag{3.6}$$

which can be expressed more compactly as

$$E_{GL}=\tfrac{1}{2}\int b\,\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x\mathrm{d}y=\tfrac{1}{2}\int\boldsymbol{u}\cdot\mathscr{L}(b)\boldsymbol{u}\,\mathrm{d}x\mathrm{d}y, \tag{3.7}$$

where $\boldsymbol{v}$ is defined by (the dimensional version of) (1.3) and $\nabla\cdot(b\boldsymbol{u})=0$ has been used.

Second, the GL equations (2.42) have a natural vorticity Ω which is the two-dimensional curl of $\boldsymbol{v}$, defined by

$$\Omega \equiv \nabla\wedge\boldsymbol{v} \equiv \partial_x v_2 - \partial_y v_1 \,. \tag{3.8}$$

By taking the curl of the GL motion equation (2.42), one finds that the vorticity Ω is locally conserved, as

$$\partial_t \Omega + \nabla\cdot(\boldsymbol{u}\Omega) = 0\,, \tag{3.9}$$

and that the potential vorticity Ω/b is convected, as

$$\Big(\partial_t + \boldsymbol{u}\cdot\nabla\Big)\left(\frac{\Omega}{b}\right) = 0\,. \tag{3.10}$$

This last equation gives the vorticity stretching relation along flow lines of the GL equations: when the mean fluid depth b changes, the vorticity Ω seen by a fluid parcel changes in proportion. The convection of Ω/b combined with the weighted divergence condition (2.42) and the no-flux boundary condition $\widehat{\boldsymbol{n}}\cdot\boldsymbol{u} = 0$ yields an infinity of conservation laws in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int b\,\Phi\!\left(\frac{\Omega}{b}\right)\,\mathrm{d}x\mathrm{d}y = 0\,, \tag{3.11}$$

for any differentiable function Φ.

The GL equations (2.42) also possess a Kelvin circulation theorem. Specifically, for any closed curve $\gamma(t)$ moving with the fluid, the transport theorem (cf. Batchelor 1967, p. 273), the Leibnitz identity $\nabla(\boldsymbol{u}\cdot\boldsymbol{v}) = (\nabla\boldsymbol{u})\boldsymbol{v} + (\nabla\boldsymbol{v})\boldsymbol{u}$, and equation (2.42) imply that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma(t)} \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x} &= \oint_{\gamma(t)} \big[\partial_t\boldsymbol{v} + \boldsymbol{u}\cdot\nabla\boldsymbol{v} - \big(\nabla\boldsymbol{v}\big)\boldsymbol{u}\big]\cdot\mathrm{d}\boldsymbol{x} \\ &= -\oint_{\gamma(t)} \nabla\Big(gh - \tfrac{1}{2}|\boldsymbol{u}|^2 + \boldsymbol{u}\cdot\boldsymbol{v}\Big)\cdot\mathrm{d}\boldsymbol{x} = 0\,.\end{aligned} \tag{3.12}$$

An application of the Stokes theorem to the left-hand side of (3.12) shows that the GL vorticity Ω given by (3.8) is conserved on fluid parcels:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} \Omega\,\mathrm{d}x\mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}t}\oint_{\gamma(t)} \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x} = 0\,, \tag{3.13}$$

where $\Gamma(t)$ is the region enclosed by the curve $\gamma(t)$ moving with the fluid.

Note that the conserved energy E_{GL} is a positive-definite quadratic form (3.6) while the potential vorticity Ω/b, being convected (3.10), is uniformly bounded by its initial data. These combined features distinguish the GL (and lake) equations from other shallow-water equations such as (2.30) and the Green–Naghdi equations (discussed in §5). Indeed, global well-posedness has recently been established for both the lake (1.1) and GL (2.42) equations by combining these energy and vorticity estimates (Levermore *et al.* 1996*a*, *b*). However, no such result exists for either the shallow water equations (2.30) or the Green–Naghdi equations.

3.3. *GL Hamilton's principle*

The GL equations (2.42) arise from Hamilton's principle with the following constrained action, which reduces to the kinetic energy (3.6) when evaluated on the constraint manifold $\eta - b = 0$:

$$\begin{aligned} A_{GL} &= \tfrac{1}{2}\int \mathrm{d}t \int \mathrm{d}x\mathrm{d}y \left[\eta|\boldsymbol{u}|^2 + \tfrac{1}{3}\eta^3(\nabla\cdot\boldsymbol{u})^2 + \eta^2(\nabla\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\nabla b) + \eta(\boldsymbol{u}\cdot\nabla b)^2\right] \\ &\quad - \int \mathrm{d}t \int \mathrm{d}x\mathrm{d}y\, gh\,(\eta - b) \\ &= \int \mathrm{d}t \int \mathrm{d}x\mathrm{d}y\, \left[\tfrac{1}{2}\,\boldsymbol{u}\cdot\mathscr{L}(\eta,b)\boldsymbol{u} - gh\,(\eta-b)\right]\,. \end{aligned} \tag{3.14}$$

Here the operator $\mathscr{L}(\eta,b)$ is given by

$$\mathscr{L}(\eta,b)\boldsymbol{u} = \eta\boldsymbol{u} - \tfrac{1}{3}\nabla(\eta^3\nabla\cdot\boldsymbol{u}) - \tfrac{1}{2}\nabla(\eta^2\boldsymbol{u}\cdot\nabla b) + \tfrac{1}{2}\eta^2(\nabla\cdot\boldsymbol{u})\nabla b + \eta(\boldsymbol{u}\cdot\nabla b)\nabla b\,, \tag{3.15}$$

and we have used the boundary condition $\widehat{\boldsymbol{n}}\cdot\boldsymbol{u} = 0$ in integrating by parts. In (3.14) the surface height h enters the action A_{GL} as a Lagrange multiplier which restricts the thickness of the water layer η to equal the equilibrium depth to the bottom $b(\boldsymbol{x})$. The action in (3.14) is similar to the one introduced by Miles & Salmon (1985) for the Green–Naghdi equations (see §5), but here the gravity waves have been removed entirely by imposing the constraint $\eta = b$ which sets their amplitude to zero.

In expression (3.14) for the GL action, the total fluid depth η and the velocity components u^i $(i = 1,2)$ are given in terms of partial derivatives of the Lagrangian labels, $\tilde{l}^A(\boldsymbol{x},t)$, which move with the fluid and, hence satisfy

$$\frac{\mathrm{d}\tilde{l}^A}{dt} \equiv \partial_t\tilde{l}^A + u^i\partial_i\tilde{l}^A = 0\,, \qquad A = 1,2\,, \tag{3.16}$$

where $\partial_i = \partial/\partial x^i$ for $i = 1,2$ and we sum over repeated indices. Incompressibility implies that the (unconstrained) total fluid depth η satisfies

$$\eta = \det\left(\partial_i\tilde{l}^A\right). \tag{3.17}$$

Thus, the fluid depth η is the Jacobian for the transformation from the current Eulerian position x^i to the Lagrangian label $\tilde{l}^A$ with $i, A = 1,2$. As a consequence of its definition (3.17) and the relation (3.16), the fluid depth η also obeys the continuity equation,

$$\eta_t + \nabla\cdot(\eta\boldsymbol{u}) = 0\,. \tag{3.18}$$

Consequently, the value $\eta = b(\boldsymbol{x})$ is preserved in time, provided the weighted incompressibility condition in (2.42) is satisfied. In addition, equation (3.16) implies the following relation for the horizontal components of the fluid velocity:

$$u^i = -(\tilde{D}^{-1})^i_A\partial_t\tilde{l}^A\,, \tag{3.19}$$

where the matrix $(\tilde{D}^{-1})^i_A$ is the inverse of the matrix

$$\tilde{D}^A_i \equiv \partial_i\tilde{l}^A\,. \tag{3.20}$$

The inverse matrix $(\tilde{D}^{-1})^i_A$ exists, since $\det(\tilde{D}^A_i) = \eta \neq 0$, when the constraint is imposed that $\eta = b \neq 0$. Upon using the definitions (3.17) and (3.19) of η and $\boldsymbol{u}$ in terms of $\tilde{l}^A$, and the boundary condition $\widehat{\boldsymbol{n}}\cdot\boldsymbol{u} = 0$, the GL equations (2.42) result from stationarity of the action A_{GL} in (3.14), under variations with respect to Lagrangian fluid labels $\tilde{l}^A(\boldsymbol{x},t)$ at fixed Eulerian position and time.

4. Relation to layer means of the Euler model

4.1. *Averaged energy, vorticity and circulation conservation*

The local conservation laws for energy (3.5) and vorticity (3.9) for the GL equations are inherited from the three-dimensional Euler equations (2.5)–(2.7) with boundary conditions (2.8)–(2.11). More specifically, they correspond to exact local conservation laws satisfied by vertically averaged Euler quantities upon applying the rigid-lid and long-wave approximations.

First consider energy. If we choose f to be the Euler energy density $\frac{1}{2}(|\boldsymbol{u}|^2 + \delta^2 w^2)$ in (2.14), we obtain

$$\partial_t \left[\eta \tfrac{1}{2}\overline{(|\boldsymbol{u}|^2 + \delta^2 w^2)}\right] + \nabla \cdot \left[\eta \tfrac{1}{2}\overline{(|\boldsymbol{u}|^2 + \delta^2 w^2)\boldsymbol{u}}\right] = -\eta\, \overline{(\boldsymbol{u} \cdot \nabla p + w\partial_z p)}. \tag{4.1}$$

Moreover, it follows from incompressibility (2.7), the definition (2.13) of layer means and the boundary conditions (2.8)–(2.10) that

$$\eta\, \overline{(\boldsymbol{u} \cdot \nabla p + w\partial_z p)} = \partial_t \left[\tfrac{1}{2}\epsilon^2 h^2\right] + \nabla \cdot [\eta \overline{p\boldsymbol{u}}]\,, \tag{4.2}$$

whereby we obtain the local conservation law for layer mean energy

$$\partial_t \left[\eta \tfrac{1}{2}\overline{(|\boldsymbol{u}|^2 + \delta^2 w^2)} + \tfrac{1}{2}\epsilon^2 h^2\right] + \nabla \cdot \left[\eta\, \left(\tfrac{1}{2}\overline{(|\boldsymbol{u}|^2 + \delta^2 w^2)\boldsymbol{u}} + \overline{p\boldsymbol{u}}\right)\right] = 0\,. \tag{4.3}$$

Here the quantities $\boldsymbol{u}$, w, p and h solve the non-dimensional Euler equations (2.5)–(2.7) with the boundary conditions (2.8)–(2.11).

The local conservation law for layer mean energy satisfied when $\boldsymbol{u}$, w, p and h solve the rigid-lid boundary approximation is obtained from (4.3) by formally setting $\epsilon = 0$ and replacing η with b. This yields

$$\partial_t \left[b \tfrac{1}{2}\overline{(|\boldsymbol{u}|^2 + \delta^2 w^2)}\right] + \nabla \cdot \left[b\, \left(\tfrac{1}{2}\overline{(|\boldsymbol{u}|^2 + \delta^2 w^2)\boldsymbol{u}} + \overline{p\boldsymbol{u}}\right)\right] = 0\,, \tag{4.4}$$

where the overbar is now understood to mean the rigid-lid average (2.22). Because the $\boldsymbol{u}$, w, p and h that solve the rigid-lid approximation are formally within $O(\epsilon^2)$ of those that solve the free-boundary problem, the corresponding energy densities and fluxes will agree to within $O(\epsilon^2)$.

We furthermore consider δ to be small and assume the leading-order horizontal velocity is columnar, just as we did in the derivation of the GL equations. If we approximate $\overline{|\boldsymbol{u}|^2}$ as we did with (2.35) and $\overline{w^2}$ by its leading order (2.29), the energy density of (4.4) reduces to

$$\tfrac{1}{2}\overline{(|\boldsymbol{u}|^2 + \delta^2 w^2)} = \tfrac{1}{2}\, |\overline{\boldsymbol{u}}|^2 + \delta^2 \tfrac{1}{6}\, (\overline{\boldsymbol{u}} \cdot \nabla b)^2 + O(\delta^4)\,. \tag{4.5}$$

By arguing similarly while using (2.32)–(2.33) to approximate $\overline{p\boldsymbol{u}}$, the flux of (4.4) is expressible as

$$\tfrac{1}{2}\overline{(|\boldsymbol{u}|^2 + \delta^2 w^2)\boldsymbol{u}} + \overline{p\boldsymbol{u}} = \overline{\boldsymbol{u}}\Big(h - \tfrac{1}{2}|\overline{\boldsymbol{u}}|^2 + \overline{\boldsymbol{u}} \cdot \overline{\boldsymbol{v}}\Big) + \delta^2 \tfrac{1}{6} b^2 \overline{\boldsymbol{u}} \partial_t (\nabla \cdot \overline{\boldsymbol{u}}) + O(\delta^4)\,, \tag{4.6}$$

where $\overline{\boldsymbol{v}}$ is defined in (2.40). Because $\overline{\boldsymbol{u}}$ and $\overline{\boldsymbol{v}}$ are formally within $O(\delta^4)$ of the $\boldsymbol{u}$ and $\boldsymbol{v}$ that solve the GL equations, relations (4.5) and (4.6) show that the density and flux of (4.4) agree to within $O(\delta^4)$ with the non-dimensional form of the density and flux of (3.5). In particular, this argument shows that the total energy for the rigid-lid approximation agrees with E_{GL} in equation (3.6) to within $O(\delta^4)$. By taking into account the corrections of $O(\epsilon^2)$ for the free-boundary problem and the relative ordering (2.17), we can then conclude that the total energy of the original Euler equations agrees with E_{GL} to within $o(\delta^2)$.

The three-dimensional Euler equations also possess a local conservation law for a vertically averaged z-dependent component of the vorticity. To express this conservation law we introduce the non-dimensional form of the horizontal vorticity field $\boldsymbol{\omega}$ by

$$\boldsymbol{\omega} \equiv \delta^2 \begin{pmatrix} \partial_y w \\ -\partial_x w \end{pmatrix} - \begin{pmatrix} \partial_z u_2 \\ -\partial_z u_1 \end{pmatrix}. \tag{4.7}$$

One can derive the remarkable local conservation law

$$\partial_t \left[\overline{\nabla \wedge \boldsymbol{u}} - \frac{1}{\eta}\left(\nabla \eta \cdot \overline{z\boldsymbol{\omega}} + \epsilon^2 (b\nabla h - h\nabla b) \cdot \overline{\boldsymbol{\omega}} \right) \right] + \nabla \cdot \boldsymbol{J} = 0, \tag{4.8}$$

where the flux $\boldsymbol{J}$ is given by

$$\begin{aligned} \boldsymbol{J} \equiv\ & \overline{\boldsymbol{u}\nabla \wedge \boldsymbol{u}} - \overline{\boldsymbol{\omega} w} \\ & -\frac{1}{\eta}\left[\left(\overline{z\boldsymbol{u}\boldsymbol{\omega}} - \overline{z\boldsymbol{\omega}\boldsymbol{u}} \right) \nabla \eta + \epsilon^2 \left(\overline{\boldsymbol{u}\boldsymbol{\omega}} - \overline{\boldsymbol{\omega}\boldsymbol{u}} \right)(b\nabla h - h\nabla b) + \epsilon^2 \partial_t h \left(\overline{(z+b)\boldsymbol{\omega}} \right) \right]. \end{aligned} \tag{4.9}$$

This result is a particular case of a more general result for vector local conservation laws with skew-symmetric flux tensors studied by Camassa & Levermore (1997). The conserved density in (4.8) is the vertical average of the inner product of the three-dimensional vorticity $(\boldsymbol{\omega}, \nabla\wedge\boldsymbol{u})$ and the vector field that linearly interpolates between the normal vector $(-\epsilon^2\nabla h, 1)$ at the top surface $z = h$ and the normal vector $(\nabla b, 1)$ at the basin bottom $z = -b$.

The analogous local conservation law satisfied when $\boldsymbol{u}$, w, p and h solve the rigid-lid boundary approximation is obtained from (4.8) by formally setting $\epsilon = 0$ and replacing η with b. This yields

$$\partial_t \left(\overline{\nabla\wedge\boldsymbol{u}} - \frac{1}{b}\nabla b \cdot \overline{z\boldsymbol{\omega}} \right) + \nabla \cdot \left[\overline{\boldsymbol{u}\nabla\wedge\boldsymbol{u}} - \overline{\boldsymbol{\omega} w} - \frac{1}{b}\left(\overline{z\boldsymbol{u}\boldsymbol{\omega}} - \overline{z\boldsymbol{\omega}\boldsymbol{u}} \right)\nabla b \right] = 0. \tag{4.10}$$

When we again consider δ to be small and assume the leading-order horizontal velocity is columnar, one finds for $\overline{\nabla\wedge\boldsymbol{u}}$

$$\overline{\nabla\wedge\boldsymbol{u}} = \nabla\wedge\bar{\boldsymbol{u}} - \frac{1}{b}\nabla b \wedge \left(\boldsymbol{u}\big|_{z=-b} - \bar{\boldsymbol{u}} \right) \tag{4.11}$$

$$= \nabla\wedge\bar{\boldsymbol{u}} - \frac{1}{b}\nabla b \wedge \overline{z\partial_z \boldsymbol{u}} \tag{4.12}$$

$$= \nabla\wedge\bar{\boldsymbol{u}} - \delta^2 \frac{1}{b}\nabla b \wedge \overline{z\nabla w} + \frac{1}{b}\nabla b \cdot \overline{z\boldsymbol{\omega}}, \tag{4.13}$$

where the last step uses the definition (4.7) of $\boldsymbol{\omega}$. The density of (4.10) can then be shown to satisfy

$$\begin{aligned} \overline{\nabla\wedge\boldsymbol{u}} - \frac{1}{b}\nabla b \cdot \overline{z\boldsymbol{\omega}} &= \nabla\wedge\bar{\boldsymbol{u}} + \delta^2 \tfrac{1}{3}\, b\, \nabla b \wedge \nabla(\nabla \cdot \bar{\boldsymbol{u}}) + O(\delta^4) \\ &= \nabla\wedge\bar{\boldsymbol{v}} + O(\delta^4). \end{aligned} \tag{4.14}$$

Similarly, the flux of (4.10) satisfies

$$\overline{\boldsymbol{u}\nabla\wedge\boldsymbol{u}} - \overline{\boldsymbol{\omega} w} - \frac{1}{b}\left(\overline{z\boldsymbol{u}\boldsymbol{\omega}} - \overline{z\boldsymbol{\omega}\boldsymbol{u}} \right)\nabla b = \bar{\boldsymbol{u}}\nabla\wedge\bar{\boldsymbol{v}} + O(\delta^4). \tag{4.15}$$

These relations show that the density and flux of (4.8) are within $O(\delta^4)$ of the non-dimensional form of the density and flux of (3.9). Hence, the $\boldsymbol{v}$ of the GL equations, which deviates from $\boldsymbol{u}$ by the $O(\delta^2)$ correction for hydrostatic imbalance in (1.3), can be understood as the vector whose curl represents the locally conserved density of (4.8) to $O(\delta^4)$ for the rigid-lid approximation. By taking into account the corrections of $O(\epsilon^2)$ for the free-boundary problem and the relative ordering (2.17), the above approximation to the density of (4.8) for the original Euler equations is of $o(\delta^2)$.

4.2. *Averaged Euler Hamilton's principle for columnar motion*

The non-dimensional Euler equations in three dimensions, (2.5)–(2.6), can be derived from an action principle – namely, they characterize the critical points of an action A_{Euler} of the form

$$A_{Euler} = \int \mathrm{d}t \int \mathrm{d}x\mathrm{d}y \int_{-b}^{\epsilon^2 h} \mathrm{d}zD\left[\tfrac{1}{2}(|\boldsymbol{u}|^2 + \delta^2 w^2) - z - p(D^{-1} - 1)\right] . \tag{4.16}$$

Here $D = \det(D_i^A)$, where $D_i^A = (\partial l^A/\partial x^i)$ is the 3×3 Jacobian matrix for the map from Eulerian coordinates to Lagrangian fluid labels, $l^A(\boldsymbol{x}, z, t), A = 1, 2, 3$. These Lagrangian labels specify the fluid particle currently occupying Eulerian position $(x^1, x^2, x^3) = (\boldsymbol{x}, z)$. They satisfy the advection law, $0 = \mathrm{d}l^A/\mathrm{d}t = \partial l^A/\partial t + u^i D_i^A + wD_3^A$, thereby determining the velocity components $(\boldsymbol{u}, w)$ in the action principle, as

$$\left.\begin{aligned} u^i &= -(D^{-1})^i_A \partial_t l^A , \qquad i = 1, 2, \\ w &= -(D^{-1})^3_A \partial_t l^A , \end{aligned}\right\} \tag{4.17}$$

where, as usual, we sum on repeated indices. Variations in (4.16) with respect to l^A yield the non-dimensional Euler equations for kinematic boundary conditions (2.9)–(2.11). The constraint $D = 1$ imposed by the Lagrange multiplier p (the pressure) implies incompressibility. For more details, see e.g. Holm, Marsden & Ratiu (1986) and Miles & Salmon (1985).

Following Miles & Salmon (1985), we restrict the action principle (4.16) to variations among solutions of the following form for the Lagrangian labels (this restriction is equivalent to the columnar motion ansatz in equation (2.28)):

$$\left.\begin{aligned} l^A &= \tilde{l}^A(\boldsymbol{x}, t), \qquad A = 1, 2, \\ l^3 &= \tilde{l}^3 = \frac{z + b}{\epsilon^2 h + b} \equiv \frac{z + b}{\eta(\boldsymbol{x}, t)}, \end{aligned}\right\} \tag{4.18}$$

from which (4.17) implies $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ and, hence, $w = -z\nabla \cdot \boldsymbol{u} - \nabla \cdot (b\boldsymbol{u})$. For restricted solutions of this type, $D = \overline{D} = \tilde{D}\tilde{l}^3_{,3} = \tilde{D}/\eta$; so performing the vertical integrations reduces the action (4.16) to

$$\begin{aligned} A_{GN} = \tfrac{1}{2} \int \mathrm{d}t \int \mathrm{d}x\mathrm{d}y \left[\eta|\boldsymbol{u}|^2 + \delta^2\left(\tfrac{1}{3}\eta^3(\nabla \cdot \boldsymbol{u})^2 + \eta^2(\nabla \cdot \boldsymbol{u})(\boldsymbol{u} \cdot \nabla b) + \eta(\boldsymbol{u} \cdot \nabla b)^2\right)\right] \\ -\tfrac{1}{2} \int \mathrm{d}t \int \mathrm{d}x\mathrm{d}y\ \eta(\eta - 2b) + \int \mathrm{d}t \int \mathrm{d}x\mathrm{d}y\ \overline{p}(\tilde{D} - \eta), \end{aligned} \tag{4.19}$$

where $\eta = b + \epsilon^2 h = \tilde{D}$ on the constraint manifold, $\tilde{D} = \det\left(\partial_i \tilde{l}^A\right)$ and $\overline{p}$ is the layer mean of p. The kinetic energy terms in the action A_{GN} agree with the layer mean relation (4.5).

For such columnar solutions, advection of the fluid labels implies

$$\partial_t \tilde{l}^A = -\boldsymbol{u} \cdot \nabla \tilde{l}^A \quad \Rightarrow \quad \boldsymbol{u} = -\frac{\partial \boldsymbol{x}}{\partial \tilde{l}^A} \partial_t \tilde{l}^A, \quad i, A = 1, 2, \tag{4.20}$$

and incompressibility implies $\eta = \tilde{D}$, cf. equation (3.17). Consequently, we may take variations of A_{GN} with respect to $\tilde{l}^A$ at fixed x, y and t, by using the chain rule for variational derivatives.

Neglecting terms of $O(\epsilon^2)$ in A_{GN} gives

$$A_{GL} = \tfrac{1}{2}\int \mathrm{d}t \int \mathrm{d}x\mathrm{d}y \left[\tilde{D}|\boldsymbol{u}|^2 + \delta^2\left(\tfrac{1}{3}\tilde{D}^3(\nabla\cdot\boldsymbol{u})^2 + \tilde{D}^2(\nabla\cdot\boldsymbol{u})(\boldsymbol{u}\cdot\nabla b) + \tilde{D}(\boldsymbol{u}\cdot\nabla b)^2\right)\right] + \int \mathrm{d}t \int \mathrm{d}x\mathrm{d}y\,\overline{p}\,(\tilde{D} - b) + O(\epsilon^2). \tag{4.21}$$

Taking variations of A_{GL} with respect to $\tilde{l}^A$ at fixed x, y and t gives the GL equations (2.42) with $h = \overline{p}$. Thus, the GL equations (2.42) extremize the layer mean kinetic energy subject to the rigid-lid constraint, $\tilde{D} = b$, which is imposed by the layer mean pressure as a Lagrange multiplier.

5. Green–Naghdi equations

This Section discusses the relation between the GL equations (2.42) and the Green–Naghdi (GN) equations (Green & Naghdi 1976) for nonlinearly dispersive gravity waves in shallow water. The GN equations are (in dimensional form)

$$\left.\begin{aligned} \partial_t\boldsymbol{u} &= -\boldsymbol{u}\cdot\nabla\boldsymbol{u} - g\nabla(\eta - b) + \frac{1}{\eta}\nabla\left(\eta^2\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right) - \left(\frac{\mathrm{d}\beta}{\mathrm{d}t}\right)\nabla b,\\ \partial_t\eta &= -\nabla\cdot(\eta\boldsymbol{u}), \end{aligned}\right\} \tag{5.1}$$

where $\eta(\boldsymbol{x}, t) = b(\boldsymbol{x}) + h(\boldsymbol{x}, t)$ is the local depth of the water. The quantities α and β are given by

$$\left.\begin{aligned} \alpha &= \tfrac{1}{3}\eta\nabla\cdot\boldsymbol{u} + \tfrac{1}{2}\boldsymbol{u}\cdot\nabla b = -\frac{1}{3}\frac{\mathrm{d}\eta}{\mathrm{d}t} + \frac{1}{2}\frac{\mathrm{d}b}{\mathrm{d}t},\\ \beta &= \tfrac{1}{2}\eta\nabla\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\nabla b = -\frac{1}{2}\frac{\mathrm{d}\eta}{\mathrm{d}t} + \frac{\mathrm{d}b}{\mathrm{d}t}, \end{aligned}\right\} \tag{5.2}$$

and $\mathrm{d}/\mathrm{d}t = \partial_t + \boldsymbol{u}\cdot\nabla$ is the material derivative following the horizontal velocity, $\boldsymbol{u}$. The GN equations provide a vertically averaged description of shallow-water motion with a free surface under gravity. These equations were derived in the setting of one horizontal dimension and a flat bottom by Su & Gardner (1969) as a dispersive correction to the usual shallow-water equations. They were derived in the more general setting used here by Green & Naghdi (1976) by requiring the incompressible columnar motion to satisfy conservation of energy and invariance under rigid body translations. They were rediscovered by Bazdenkov, Morozov & Pogutse (1985), who also considered the case of a rotating frame. Finally, they were derived from Hamilton's principle with action (4.19) by Miles & Salmon (1985).

The GN equations retain finite-amplitude gravity waves and their associated Boussinesq-type dispersion properties, which are neglected in the GL equations upon taking the small-amplitude limit ($\epsilon^2 \to 0$). We show here that the GL equations can be understood as the small-wave-amplitude limit of the GN equations. In fact, we could have derived the GN equations as an intermediate step in our asymptotic expansion for the Euler equations in §2, by *not* imposing small Froude number (and, thus, small wave amplitude). The GN equations possess local conservation laws of energy and vorticity which, in complete analogy with the GL equations, are manifestations of the layer mean equations (4.3) and (4.8). In addition, the similarity of the action A_{GN} in (4.19) for the GN equations to the action A_{GL} in equation (3.14) or (4.21) for the GL equations also makes it possible to transfer many of the structural results of §3 for GL immediately over to the GN case. For example, see Camassa *et al.* (1996) for a parallel description of the Hamiltonian formulations of these two sets of equations.

The global well-posedness of classical solutions for the GL equations has recently been established by Levermore & Oliver (1997) and Oliver (1997*a*). This result provides a foundation for a rigorous justification of the formal derivation of the GL equations given in the previous Section. Indeed, Oliver (1997*b*) gives a rigorous justification of the derivation of the lake equations from the rigid-lid Euler equations for the case of periodic lateral boundary conditions. From our viewpoint, it is worthwhile to make the connection between the GL equations and the GN equations, because the global well-posedness of the GL equations validates the use of the GN equations over long times, in the limit of small wave amplitude. No such result exists for the GN equations. For finite wave amplitudes, a numerical comparison of the GN equations with the Euler equations for finite-wave amplitude free-surface incompressible flow over bottom topography has recently been discussed by Nadiga, Margolin & Smolarkiewicz (1996). They show that numerical simulations of the GN equations tend to be faithful representations of the corresponding Euler solutions, so long as wave breaking does not occur.

The GN equations (5.1)–(5.2) can assume a different form, one which is more natural from a variational standpoint and which is similar to that of the GL equations (2.42). Using the operator $\mathscr{L}(\eta, b)$ defined in (3.15) and introducing the auxiliary field

$$\eta \boldsymbol{v}_{GN} \equiv \mathscr{L}(\eta, b)\boldsymbol{u}, \tag{5.3}$$

the first equation in the GN system (5.1)–(5.2) can be rewritten (by using the second equation in (5.1) and judiciously differentiating by parts) in the form

$$\begin{aligned} &\partial_t \boldsymbol{v}_{GN} + \boldsymbol{u}\cdot\nabla\boldsymbol{v}_{GN} + (\nabla\boldsymbol{u})\boldsymbol{v}_{GN} \\ &\qquad + \nabla\left(g(\eta - b) - \tfrac{1}{2}|\boldsymbol{u}|^2 - \tfrac{1}{2}\big(\eta\nabla\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\nabla b\big)^2\right) = 0. \end{aligned} \tag{5.4}$$

The GL equations (2.42) now follow immediately from this form of the GN equations by formally taking the asymptotic limit $\epsilon \to 0$ after the rescaling in which

$$\nabla \mapsto \nabla, \qquad \partial_t \mapsto \epsilon\,\partial_t, \qquad \boldsymbol{u} \mapsto \epsilon\,\boldsymbol{u}, \qquad \eta \mapsto b + \epsilon^2 h, \tag{5.5}$$

so that

$$\begin{aligned} &g(\eta - b) \mapsto \epsilon^2 g h, \qquad \mathscr{L}(\eta, b) \mapsto \mathscr{L}(b) + O(\epsilon^2), \\ &\big(\eta\nabla\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\nabla b\big)^2 \mapsto \big(\nabla\cdot(b\boldsymbol{u})\big)^2 + O(\epsilon^2). \end{aligned}$$

The rescaled GN equations in this small-Froude-number limit reduce to the system

$$\left.\begin{aligned} \partial_t \boldsymbol{v}_{GN} + \boldsymbol{u}\cdot\nabla\boldsymbol{v}_{GN} + (\nabla\boldsymbol{u})\boldsymbol{v}_{GN} + \nabla\Big(gh - \tfrac{1}{2}|\boldsymbol{u}|^2\Big) = O(\epsilon^2), \\ \nabla\cdot(b\boldsymbol{u}) = O(\epsilon^2), \qquad b\boldsymbol{v}_{GN} = \mathscr{L}(b)\boldsymbol{u} + O(\epsilon^2), \end{aligned}\right\} \tag{5.6}$$

where $\mathscr{L}(b)$ is given in equation (3.1). When the $O(\epsilon^2)$ terms are dropped, we see that this asymptotic limit of the GN equations (5.1), which corresponds to looking at small surface height displacements over long time scales, reduces to the GL equations (2.42). Notice that, in complete analogy with the GL equations, the auxiliary field $\boldsymbol{v}_{GN}$ can be interpreted as the vector whose curl represents the locally conserved density (4.8) to $O(\delta^4)$ (see Camassa *et al.* 1996 for further discussion).

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